Relative Regular Objects in Categories

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Abstract

We define the concept of a regular object with respect to another object in an arbitrary category. We present basic properties of regular objects and we study this concept in the special cases of abelian categories and locally finitely generated Grothendieck categories. Applications are given for categories of comodules over a coalgebra and for categories of graded modules, and a link to the theory of generalized inverses of matrices is presented. Some of the techniques we use are new, since dealing with arbitrary categories allows us to pass to the dual category.

0   Introduction and preliminaries

Von Neumann regular rings play a fundamental role in Ring Theory, see [4]. A ring R is called von Neumann regular if for any $a \in R$ there exists $b \in R$ such that $a = aba$. This concept was generalized to modules in [7]. A left module M over the ring R is called regular if for any $m \in M$ there exists $g \in Hom_R(M, R)$ such that $g(m)m = m$. Basic properties of regular modules are developed in [7]. Since a morphism $f \in Hom_R(R, M)$ is uniquely given by an element $m \in M$, one can reformulate the regular condition as follows. For any $f \in Hom_R(R, M)$ there exists $g \in Hom_R(M, R)$ such that $f = f \circ g \circ f$. This suggests the definition of a more general concept of an U-regular object in a category, where U is a given object of the category. We study basic properties of regular objects in categories, with special emphasis on abelian categories and on locally finitely generated Grothendieck categories. The main source of inspiration for our results was [7]. However, even for results that sound similarly, as for example our key Theorem 2.8, stating that a finite direct sum of U-regular objects in an abelian category is also U-regular, and extending [7, Theorem 2.8], the proof is consistently different. For some results we use Mitchell Theorem to reduce to categories of modules, but most of the proofs are done inside the general abelian category. We

note that defining the concept of a relative regular object in an arbitrary category presents the advantage that we may pass to the dual category, and this is a completely new method if we compare to the techniques used when dealing with regular modules over rings. An example to illustrate this statement is Corollary 2.8, which follows from Theorem 2.7 by transferring the result to the dual category. As applications, we give some results for the category of comodules over a coalgebra and for the category of graded modules over a graded ring. We also show that there is a link between the concept of a relative regular object in a category and the theory of generalized inverses of matrices.

We refer the reader to [2] and [3] for elements of Category Theory, to [1] for facts about coalgebras, and to [6] for definitions and results about graded rings.

1   Regular objects and basic properties

Let $f : U \to M$ and $h : M \to U$ be morphisms in a category $\mathcal{A}$. We say that h is a generalized inverse of f if $f = f \circ h \circ f$ and $h = h \circ f \circ h$. It is easy to see that the morphism $f : U \to M$ has a generalized inverse if and only if there exists a morphism $g : M \to U$ such that $f = f \circ g \circ f$. Indeed, an easy computation shows that $h = g \circ f \circ g$ is a generalized inverse of f.

Definition 1.1 *Let U and M be objects of a category $\mathcal{A}$. Then M is called U-regular if and only if any morphism $f : U \to M$ has a generalized inverse.*

Remark 1.2 *(1) If U and M are objects of a category $\mathcal{A}$, then M is U-regular in $\mathcal{A}$ if and only if U is M-regular in the dual category $\mathcal{A}^0$.*
(2) If we associate to a ring R a category with one object, such that the elements of the ring are the morphisms and the composition of morphisms is just the multiplication of R, then R is a von Neumann regular ring if and only if every morphism of the associated category has a generalized inverse.

Proposition 1.3 *Assume that M is U-regular in a category $\mathcal{A}$. The following assertions hold true.*
(1) If $\pi : U \to U''$ is an epimorphism in $\mathcal{A}$, then M is also U''-regular.
(2) If $i : M' \to M$ is a monomorphism in $\mathcal{A}$, then M' is U-regular.
(3) If $\mathcal{A}$ is an additive category and U' is a direct summand of U, then M is also U'-regular.

Proof: (1) Let $f : U'' \to M$ be a morphism. Since M is U-regular and $f \circ \pi : U \to M$, there exists a morphism $g : M \to U$ such that $f \circ \pi = (f \circ \pi) \circ g \circ (f \circ \pi)$. Since π is an epimorphism, we get that $f = f \circ (\pi \circ g) \circ f$. Thus M is U''-regular.
(2) Let $f : U \to M'$ be a morphism. Since M is U-regular and $i \circ f : U \to M$, there exists

$g : M \to U$ such that $i \circ f = (i \circ f) \circ g \circ (i \circ f)$. Since i is a monomorphism, we see that $f = f \circ (g \circ i) \circ f$. Thus M' is U-regular.

(3) Let $i : U' \to U$ and $\pi : U \to U'$ be morphisms such that $\pi \circ i = 1_{U'}$. If $f : U' \to M$ is a morphism, then $f \circ \pi : U \to M$, so there exists $g : M \to U$ such that $f \circ \pi = (f \circ \pi) \circ g \circ (f \circ \pi)$. Then $f \circ \pi \circ i = (f \circ \pi) \circ g \circ (f \circ \pi) \circ i$, showing that $f = f \circ (\pi \circ g) \circ f$. Thus M is U'-regular. ∎

Proposition 1.4 *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor which is full and faithful, and let U, M be objects of $\mathcal{A}$. Then M is U-regular if and only if $F(M)$ is $F(U)$-regular.*

Proof: Assume that M is U-regular. Let $\overline{f} : F(U) \to F(M)$. Then there exists $f : U \to M$ such that $\overline{f} = F(f)$. Since M is U-regular, there exists $g : M \to U$ such that $f = f \circ g \circ f$. Then $\overline{f} = F(f) \circ F(g) \circ F(f) = \overline{f} \circ F(g) \circ \overline{f}$. Thus $F(M)$ is $F(U)$-regular.

For the converse, assume that $F(M)$ is $F(U)$-regular and let $f : U \to M$. Then $F(f) : F(U) \to F(M)$, so there exists $\overline{g} : F(M) \to F(U)$ such that $F(f) = F(f) \circ \overline{g} \circ F(f)$. Now there is $g : M \to U$ such that $\overline{g} = F(g)$, and then we see that $f = f \circ g \circ f$, showing that M is U-regular. ∎

Example 1.5 *Let M and U be sets. Then M is U-regular in the category Set of sets. Indeed, let $f : U \to M$ be a map. For any $m \in Im(f)$ fix $a_m \in U$ such that $m = f(a_m)$. Also fix some element $a \in U$, and define the map $g : M \to U$ such that $g(m) = a_m$ for any $m \in Im(f)$, and $g(m) = a$ for any $m \notin Im(f)$. Then $f(g(f(u))) = f(a_{f(u)}) = f(u)$ for any $u \in U$, so $f = f \circ g \circ f$.*

Remark 1.6 *(1) The definition of regular objects shows immediately that if U is an object of the additive category $\mathcal{A}$, then U is U-regular if and only if the endomorphism ring $End(U)$ is a regular ring.*

(2) If M is U-regular, then any epimorphism $f : U \to M$ splits, i.e. there exists a morphism $g : M \to U$ such that $f \circ g = Id_M$.

2 Regular objects in abelian categories

In this section we study regular objects in abelian categories. A key characterization is the following.

Proposition 2.1 *Let M and U be objects of an abelian category $\mathcal{A}$. Then M is U-regular if and only if $Ker(f)$ is a direct summand of U and $Im(f)$ is a direct summand of M for any morphism $f : U \to M$.*

Proof: Assume that M is U-regular, and let $f : U \rightarrow M$ be a morphism. Then there exists a morphism $g : M \rightarrow U$ such that $f = f \circ g \circ f$. Let $f' : U \rightarrow Im(f)$ be the corestriction of f and $j : Im(f) \rightarrow M$ be the inclusion morphism. Thus $f = j \circ f'$. Hence $j \circ f' = j \circ f' \circ g \circ j \circ f'$. Since j is a monomorphism we have that $f' = f' \circ g \circ j \circ f'$. But f' is an epimorphism, so $(f' \circ g) \circ j = Id_{Im(f)}$, showing that $Im(f)$ is a direct summand in M.

On the other hand $f' \circ (g \circ j) = Id_{Im(f)}$ shows that $Ker(f') = Ker(f)$ is a direct summand in U.

For the converse, to show that M is U-regular, let $f : U \rightarrow M$ be a morphism. As above let $f' : U \rightarrow Im(f)$ be the corestriction of f and $j : Im(f) \rightarrow M$ be the inclusion morphism. Since $Ker(f)$ is a direct summand of U, there exists $\beta : Im(f) \rightarrow U$ such that $f' \circ \beta = Id_{Im(f)}$. Also, since $Im(f)$ is a direct summand in M, there exists $\alpha : M \rightarrow Im(f)$ such that $\alpha \circ j = Id_{Im(f)}$. Define $g : M \rightarrow U$ by $g = \beta\alpha$. Then

$$
\begin{aligned}
f \circ g \circ f &= f \circ \beta \circ \alpha \circ f \\
&= j \circ f' \circ \beta \circ \alpha \circ j \circ f' \\
&= j \circ f' \\
&= f
\end{aligned}
$$

We conclude that M is U-regular. ∎

Corollary 2.2 *Let M and U be objects of an abelian category $\mathcal{A}$. The following assertions hold.*
(1) If U is injective, then M is U-regular if and only if $Im(f)$ is a direct summand of M for any morphism $f : U \rightarrow M$.
(2) If M is projective, then M is U-regular if and only if $Ker(f)$ is a direct summand of U for any morphism $f : U \rightarrow M$.

Corollary 2.3 *If $\mathcal{A}$ is a semisimple abelian category, then M is U-regular for any objects M and U of $\mathcal{A}$. Otherwise stated, every morphism in a semisimple abelian category has a generalized inverse.*

Remark 2.4 *As the referee showed us, the concept of relative regular object in a category can be related to the theory of generalized inverses of matrices. Let A be a $n \times m$-matrix and B be a $m \times n$-matrix. Then A is called a generalized inverse of B if $ABA = A$ and $BAB = B$ (see for example [5, Sec. 12.7]).*
If we apply Corollary 2.3 to the category of finite dimensional vector spaces over a field, we obtain the classical result that every matrix has a generalized inverse (see [5, p. 428, Prop. 1]).

Corollary 2.5 *Let N be a subobject of an object M in an abelian category. If M is N-regular, then N is a direct summand of M.*

We are now interested to study finite direct sums of regular objects in an abelian category. The following two results will be of help for reducing the study to categories of modules.

Proposition 2.6 *([2, Proposition 11.7]) Let $\mathcal{D}_0$ be a small subcategory of an abelian category $\mathcal{D}$. Then there exists a small abelian full subcategory $\mathcal{D}'$ of $\mathcal{D}$ such that $\mathcal{D}_0$ is a subcategory of $\mathcal{D}'$.*

Theorem 2.7 *(Mitchell Theorem, see [2, Theorem 11.6]) Let $\mathcal{C}$ be a small abelian category. Then there exist a ring A and a full and faithful exact functor $H : \mathcal{C} \to (Mod - A)^0$.*

Now we can prove the main result of this section.

Theorem 2.8 *Let $M_1, M_2, \ldots, M_n$ be U-regular objects of an abelian category $\mathcal{A}$, where U is an object of $\mathcal{A}$. Then $M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is U-regular.*

Proof: Proposition 2.6, Theorem 2.7 and Proposition 1.4 show that it is enough to prove the result for the case where $\mathcal{A} = R - mod$, a category of modules over a ring R. Using an inductive argument, it is enough to prove for $n = 2$. Thus let M_1 and M_2 be R-modules which are U-regular for a certain R-module U. Let $f : U \to M_1 \oplus M_2$ be a morphism of R-modules. If $\pi_i : M_1 \oplus M_2 \to M_i$, $i = 1, 2$ are the natural projections, let $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. Then $Ker(f) = Ker(f_1) \cap Ker(f_2)$. Since M_1 is U-regular we have that $Ker(f_1)$ is a direct summand in U. Since M_2 is U-regular, then by Proposition 1.3 we see that $Ker(f_2 \circ i) = Ker(f_1) \cap Ker(f_2)$ is a direct summand of $Ker(f_1)$, where $i : Ker(f_1) \to U$ is the inclusion map. Therefore $Ker(f) = Ker(f_1) \cap Ker(f_2)$ is a direct summand of U.

Now we show that $Im(f)$ is a direct summand in $M_1 \oplus M_2$. Clearly $Im(f) \subseteq Im(f_1) + Im(f_2) = Im(f_1) \oplus Im(f_2)$. Since $Im(f_i)$ is a direct summand of M_i, $i = 1, 2$, we have that $Im(f_1) \oplus Im(f_2)$ is a direct summand of $M_1 \oplus M_2$. Thus it is enough to prove that $Im(f)$ is a direct summand of $M_0 = Im(f_1) \oplus Im(f_2)$.

The morphism π_1 induces the exact sequence

$$0 \longrightarrow Ker(\pi_1') \overset{j}{\longrightarrow} Im(f) \overset{\pi_1'}{\longrightarrow} Im(f_1) \longrightarrow 0$$

where π_1' is the restriction of π_1. We have the split exact sequence

$$0 \longrightarrow Ker(f_1) \longrightarrow U \xrightarrow{f_1} Im(f_1) \longrightarrow 0$$

Then there exists $g_1 : Im(f_1) \to U$ such that $f_1 \circ g_1 = Id_{Im(f_1)}$. Let $\alpha = f \circ g_1 : Im(f_1) \to Im(f)$. Then $\pi_1' \circ \alpha = \pi_1' \circ f \circ g_1 = f_1 \circ g_1 = Id_{Im(f_1)}$, so $Im(\alpha)$ is a direct summand of $Im(f)$.

If $\pi_1'' : M_0 \to Im(f_1)$ is the restriction of π_1, then clearly $Ker(\pi_1') \subseteq Ker(\pi_1'') = Im(f_2)$. Since $\pi_1' \circ \alpha = Id_{Im(f_1)}$, there exists $j' : Im(f) \to Im(f_1)$ such that $j' \circ j = Id_{Ker(\pi_1')}$. Define $h = j_2 \circ u \circ j' \circ f : U \to M_2$, where $j_2 : Im(f_2) \to M_2$ is the inclusion map. Since j' is an epimorphism we have that $Im(h) = u(Ker(\pi_1')) = Ker(\pi_1')$. Now since M_2 is U-regular, we have that $Im(h)$ is a direct summand of M_2. As $Im(f_2)$ is a direct summand of M_2, we see that $Im(h)$ is a direct summand of $Im(f_2)$.

We have that $Im(f) = Ker(\pi_1') \oplus Im(\alpha)$. By the above considerations we can define a morphism $\mu : Im(f_2) \to Im(f_2)$ such that $\mu^2 = \mu$ and $Im(\mu) = Ker(\pi_1')$. Define $\theta = i \circ \mu \circ \pi_2 : M_0 \to M_0$, where $i : Im(f_2) \to M_0$ is the inclusion morphism. Then

$$\begin{aligned} \theta^2 &= i \circ \mu \circ \pi_2 \circ i \circ \mu \circ \pi_2 \\ &= i \circ \mu^2 \circ \pi_2 \\ &= i \circ \mu \circ \pi_2 \\ &= \theta \end{aligned}$$

Moreover we have that $Im(\theta) = Im(\mu) = Ker(\pi_1')$.

We also define $\gamma = i_0 \circ \alpha \circ \pi_1 : M_0 \to M_0$, where $i_0 : Im(f) \to M_0$ is the inclusion map. Clearly $Im(\gamma) = Im(\alpha)$. We have that

$$\begin{aligned} \gamma^2 &= i_0 \circ \alpha \circ \pi_1 \circ i_0 \circ \alpha \circ \pi_1 \\ &= i_0 \circ \alpha \circ \pi_1' \circ \alpha \circ \pi_1 \\ &= i_0 \circ \alpha \circ \pi_1 \\ &= \gamma \end{aligned}$$

We have that $Im(\gamma) \cap Im(\theta) = Im(\alpha) \cap Ker(\pi_1') = 0$. Moreover $\gamma \circ \theta = i_0 \circ \alpha \circ \pi_1 \circ i \circ \mu \circ \pi_2 = 0$. By [7, Lemma 1.1] we have that $Im(\gamma) + Im(\theta) = Im(\gamma) \oplus Im(\theta)$ is a direct summand in M_0, and the proof is finished. ∎

By using Remark 1.2(1) we get the following.

Corollary 2.9 *Let $M, N_1, \ldots, N_n$ be objects of an abelian category $\mathcal{A}$ such that M is N_i-regular for any $1 \le i \le n$. Then M is $N_1 \oplus \ldots \oplus N_n$-regular.*

Example 2.10 *If we have an infinite family $(N_i)_{i \in I}$ of objects such that M is N_i-regular for any $i \in I$, then we do not necessarily have that M is $\oplus_{i \in I} N_i$-regular. To see this, we consider [7, Example 3.1], where it is given an example of a von Neumann regular ring R, and a left ideal J of R which is a regular R-module, while $End_R(J)$ is not a von Neumann regular ring. Let I be an infinite set such that there is an epimorphism $R^{(I)} \to J$. We claim that J is not $R^{(I)}$-regular. Indeed, if J would be $R^{(I)}$-regular, then by Proposition 1.3(1) we have that J is J-regular. Hence by Remark 1.6(1) we would get that $End_R(J)$ is von Neumann regular, a contradiction.*

We remind that an abelian category has the property $(AB3)$ if it has arbitrary direct sums.

Corollary 2.11 *Let U be a finitely generated object of an abelian category $\mathcal{A}$ with $(AB3)$, and let $(M_i)_{i \in I}$ be a family of objects of $\mathcal{A}$ such that M_i is U-regular for any $i \in I$. Then $\oplus_{i \in I} M_i$ is U-regular.*

Proof: Let $f : U \to \oplus_{i \in I} M_i$ be a morphism. Then there is a finite subset J of I such that $Im(f)$ is a subobject of $\oplus_{i \in J} M_i$. Let $f_0 : U \to \oplus_{i \in J} M_i$ be the corestriction of f. Since $\oplus_{i \in J} M_i$ is U-regular, there exists $g_0 : \oplus_{i \in J} M_i \to U$ such that $f_0 = f_0 \circ g_0 \circ f_0$. Let $g : \oplus_{i \in I} M_i \to U$ arising from the morphism g_0 and the zero morphism $0 : \oplus_{i \in I - J} M_i \to U$. Then $f = f \circ g \circ f$, and this ends the proof. ∎

We remind that in an abelian category $\mathcal{A}$ with $(AB3)$, an object M is called U-generated if there exist a set I and an epimorphism $U^{(I)} \to M$. If I is finite (and $\mathcal{A}$ is just an abelian category), then we say that M is U-finitely generated.

Corollary 2.12 *Let U be a projective object, and let M be a U-regular object of an abelian category. The following assertions hold.*
(i) If M is U-finitely generated, then M is projective.
(ii) If $\mathcal{A}$ has the property $(AB3)$ and M is U-generated, then M is an inductive limit of projective objects.

Proof: (i) Let $f : U^n \to M$ be an epimorphism. By Corollary 2.9, M is U^n-regular. Hence by Remark 1.6(2), we see that f splits, so then M is a direct summand of U^n. We conclude that M is projective.
(ii) Since there exists an epimorphism $f : U^{(I)} \to M$, we have that M is an inductive limit of subobjects that are epimorphic images of objects of the form U^n for some positive integers n. Now the result follows from (i). ∎

Lemma 2.13 *Let $\mathcal{A}$ be an abelian category, M an object of $\mathcal{A}$, and α an element of the center of $End_{\mathcal{A}}(M)$. Then there exists β in the center of $End_{\mathcal{A}}(M)$ with $\alpha = \alpha \circ \beta \circ \alpha$ if and only if $M = Im(\alpha) \oplus Ker(\alpha)$.*

Proof: It follows from [7, Lemma 3.3] and Theorem 2.7. ∎

3   Regular objects in locally finitely generated Grothendieck categories

Throughout this section $\mathcal{A}$ is a Grothendieck category which is locally finitely generated, i.e. it has a family $(U_i)_{i \in I}$ of finitely generated generators.

Definition 3.1 *An object M of $\mathcal{A}$ is called a regular object if M is U_i-regular for any $i \in I$.*

To make the definition consistent, we need the following.

Proposition 3.2 *The concept of a regular object is independent on the choice of the family of finitely generated generators of $\mathcal{A}$.*

Proof: Assume that M is U_i-regular for any $i \in I$. Let $(V_j)_{j \in J}$ be another family of finitely generated generators of $\mathcal{A}$. Let $j \in J$. Then there exist $i_1, \ldots, i_n \in I$ and an exact sequence $U_{i_1} \oplus \ldots \oplus U_{i_n} \longrightarrow V_j \longrightarrow 0$. By Corollary 2.9 we have that M is $U_{i_1} \oplus \ldots \oplus U_{i_n}$-regular. Now by Proposition 1.3 (1) we have that M is V_j-regular. ∎

Remark 3.3 *The proof of Proposition 3.2 shows that if M is a regular object of $\mathcal{A}$, then M is V-regular for any finitely generated object V of $\mathcal{A}$.*

Example 3.4 *If $\mathcal{A} = R - mod$, then a regular object in $\mathcal{A}$ is exactly a regular R-module in the sense of [7].*

The following gives some properties of regular objects.

Theorem 3.5 *Let M be a regular object of a locally finitely generated Grothendieck category $\mathcal{A}$. The following assertions hold.*
(1) The Jacobson radical $J(M)$ of M is zero.
(2) The singular subobject $Z(M)$ of M is zero.
(3) If M is noetherian or artinian, then M is semisimple.

Proof: (1) Assume that $J(M) \neq 0$. Then there exists a finitely generated non-zero subobject N of M such that $N \subseteq J(M)$. By Corollary 2.5 we see that N is a direct summand of M, so there exists P such that $M = N \oplus P$. Since N is finitely generated, there exists a proper maximal subobject N' of N. Then $N' \oplus P$ is maximal in M, so $J(M) \subseteq N' \oplus P$. Hence $J(M) = J(M) \cap (N' \oplus P) = N' \oplus (J(M) \cap P)$.

On the other hand $J(M) = J(M) \cap (N \oplus P) = N \oplus (J(M) \cap P)$, since $N \subseteq J(M)$ and the lattice of subobjects of M is modular. Thus we have that $N' \oplus (J(M) \cap P) = N \oplus (J(M) \cap P)$, which shows that $N = N'$, a contradiction. Thus we must have $J(M) = 0$.

(2) Assume that $Z(M) \neq 0$. Then there exists a non-zero subobject N of M such that $N \simeq X/Y$ for some X and Y such that Y is essential in X. Let $(U_i)_{i \in I}$ be a family of finitely generated generators of $\mathcal{A}$. Then there is $i \in I$ and a morphism $f : U_i \to X$ such that $Im(f)$ is not a subobject of Y. Let $\pi : X \to X/Y$ be the natural projection, and let $g = \pi \circ f : U_i \to X/Y$. Clearly $Ker(g) = f^{-1}(Y)$. Since Y is essential in X, we have that $f^{-1}(Y)$ is essential in U_i. Since U_i is finitely generated, we have that $Im(g)$ is also finitely generated. Now $Im(g) \subseteq X/Y \simeq N$, so $Im(g)$ is a regular object. Hence $Ker(g)$ is a direct summand of U_i. Since $Ker(g)$ is essential in U_i, it follows that $Ker(g) = U_i$, so $g = 0$, and then $Im(f) \subseteq Y$, a contradiction. This shows that $Z(M) = 0$.

(3) Let M be noetherian. Then M is finitely generated and there exists a maximal subobject N_1 of M. Since N_1 is also finitely generated, M is N_1-regular and Corollary 2.5 shows that N_1 is a direct summand of M. Let $M = N_1 \oplus S_1$, where S_1 is a simple object. In the same way starting with N_1, we find that $N_1 = N_2 \oplus S_2$ for some N_2 and a simple object S_2. We continue recurrently, and since M is noetherian, we end with $M = S_1 \oplus S_2 \oplus \ldots \oplus S_n$ for some simple objects $S_1, S_2, \ldots, S_n$. This shows that M is semisimple.

If M is artinian, let $s(M)$ be the socle of M. We have that $s(M)$ is a finite direct sum of simple objects since M is artinian. Since $s(M)$ is also regular and essential in M, we must have $s(M) = M$, showing that M is semisimple. ∎

Proposition 3.6 *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. The following assertions are equivalent.*
(1) $\mathcal{A}$ is semisimple.
(2) Any object M of $\mathcal{A}$ is regular.

Proof: Let $i \in I$ and X be an essential subobject of U_i. Then U_i/X is U_i-regular, so $X = Ker(\pi)$ is a direct summand of U_i, where $\pi : U_i \to U_i/X$ is the natural projection. Hence $X = U_i$, so the associated Goldie torsion theory $\mathcal{G}$ is 0. But $\mathcal{A}/\mathcal{G}$ is a spectral category, so $\mathcal{A}$ is semisimple. ∎

Theorem 3.7 *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category, and let M be a regular object of $\mathcal{A}$. Then $Z(End_{\mathcal{A}}(M))$ is a regular ring.*

Proof: Since M is a regular object, M is N-regular for any finitely generated object N. Fix some f in the center of $End_{\mathcal{A}}(M)$. Let M' be a finitely generated subobject of M, and let $i : M' \to M$ be the inclusion morphism. Since M is M'-regular and $f \circ i : M' \to M$, there exists $g : M \to M'$ such that $f \circ i = (f \circ i) \circ g \circ (f \circ i)$. Since f is in the center of $End_{\mathcal{A}}(M)$, we have that $f \circ (i \circ g) = (i \circ g) \circ f$, so then

$$f \circ i = f^2 \circ (i \circ g \circ i) \tag{1}$$

and also

$$f \circ i = (i \circ g) \circ f^2 \circ i \tag{2}$$

Using equation (1) we see that $f(M') = f(f(i \circ g \circ i)(M'))$, so $M' \subseteq Im(f) + Ker(f)$. To see this, note that in general $f(X) = f(Y)$ implies that $X \subseteq Y + Ker(f)$ and $Y \subseteq X + Ker(f)$.

Since M is the union of its finitely generated subobjects, we get that $M \subseteq Im(f) + Ker(f)$, hence $M = Im(f) + Ker(f)$. We show that $Im(f) \cap Ker(f) = 0$. Indeed, if $K = Im(f) \cap Ker(f)$ were non-zero, then $L = f^{-1}(K)$ is non-zero. Since $\mathcal{A}$ is locally finitely generated, there exists a non-zero subobject N of L such that $0 \neq f(N) \subseteq K$. Since $f(K) = 0$ we get that $f^2(N) = 0$, and then by applying equation (2) we obtain that $f(N) = 0$, a contradiction. Therefore $Im(f) \cap Ker(f) = 0$ and $M = Im(f) \oplus Ker(f)$. The result follows now from Lemma 2.13. ∎

More information about the endomorphism ring of a regular object is given by the following result.

Proposition 3.8 *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category, and let M be a regular object of $\mathcal{A}$. The following assertions hold.*
(1) $End(M)$ is a semiprime ring.
(2) If M is finitely generated, then $End(M)$ is a regular ring.

Proof: (1) Let $f \in End(M)$, $f \neq 0$. Then there exists a finitely generated subobject M' of M such that $f(M') \neq 0$. By Remark 3.3 we see that M is M'-regular. Let f' be the restriction of f to M'. Then there exists $g' : M \to M'$ such that $f' = f' \circ g' \circ f'$. Let $g : M \to M$, $g = i \circ g'$, where $i : M' \to M$ is the inclusion morphism. Then the restriction of $f \circ g \circ f$ to M' is $f' \circ g' \circ f' = f' \neq 0$. Thus $f End(M) f \neq 0$, and we conclude that $End(M)$ is a semiprime ring.
(2) By Remark 3.3 we have that M is M-regular, and then $End(M)$ is a regular ring by Remark 1.6. ∎

4 Applications to coalgebras and graded rings

If C is a coalgebra over a field, we denote by $\mathcal{M}^C$ the category of right C-comodules. This is a locally finite Grothendieck category, see [1].

Theorem 4.1 *Let C be a coalgebra. Then C is a regular object in the category $\mathcal{M}^C$ if and only if C is cosemisimple.*

Proof: If C is cosemisimple then clearly C is a regular object in $\mathcal{M}^C$. Conversely, assume that C is regular in $\mathcal{M}^C$. Let M be a right comodule of finite dimension. Then there exists a monomorphism $u : M \to C^n$ for some positive integer n. Since C is regular, then C is M-regular, hence C^n is M-regular. Hence there exists a morphism $v : C^n \to M$ such that $u = u \circ v \circ u$. Since u is a monomorphism we get that $v \circ u = Id_M$, so M is isomorphic to a direct summand of C^n. This implies that M is an injective object in $\mathcal{M}^C$, and therefore the category $\mathcal{M}^C$ is semisimple, i.e. C is a cosemisimple coalgebra. ∎

Let G be a group with identity element e, and let $R = \oplus_{\sigma \in G} R_\sigma$ be a G-graded ring. Denote by $R - gr$ the category of graded left R-modules, which is a locally finitely generated Grothendieck category, see [6]. R is called von Neumann gr-regular if and only if for any $x_\sigma \in R_\sigma$ there exists $y \in R$ such that $x_\sigma = x_\sigma y x_\sigma$ (clearly we can assume that $y \in R_{\sigma^{-1}}$). It is clear that R is von Neumann gr-regular if and only if $R x_\sigma$ is a direct summand in R as an object of the category $R - gr$. If M is an object of $R - gr$ and $\sigma \in G$, the σ-suspension $M(\sigma)$ of M is the graded R-module which is equal to M as an R-module, and whose grading is given by $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$.

Theorem 4.2 *Let R be a G-graded ring. Then the following assertions are equivalent.*
(1) R is von Neumann gr-regular.
(2) $R(\sigma)$ is R-regular in the category $R - gr$ for any $\sigma \in G$.

Proof: $(1) \Rightarrow (2)$ Let $f : R \to R(\sigma)$ be a morphism in $R - gr$, and let $x_\sigma = f(1)$. Then there exists $y \in R_{\sigma^{-1}}$ such that $x_\sigma = x_\sigma y x_\sigma$. Then the map $g : R(\sigma) \to R$ defined by $g(r) = ry$ is a morphism in $R - gr$ and $f = f \circ g \circ f$.
$(2) \Rightarrow (1)$ Let $x_\sigma \in R_\sigma$. If $f : R \to R(\sigma)$ is defined by $f(r) = rx_\sigma$, then there exists a morphism $g : R(\sigma) \to R$ in $R - gr$ such that $f = f \circ g \circ f$. Then if $y = g(1)$ we obtain that $x_\sigma = x_\sigma y x_\sigma$. ∎

Corollary 4.3 *Let $R_\sigma = \oplus_{\sigma \in G} R_\sigma$ be a graded ring. Then the following assertions hold.*
(1) If R is von Neumann gr-regular then R_σ is R_e-regular for any $\sigma \in G$. In particular R is R_e-regular.
(2) If R is strongly graded and R_σ is R_e-regular for any $\sigma \in G$, then R is von Neumann gr-regular.

Proof: (1) Let $f : R_e \to R_\sigma$ be a morphism of R_e-modules, and let $x_\sigma = f(1)$. Since R is gr-regular, there exists $y \in R_{\sigma^{-1}}$ such that $x_\sigma = x_\sigma y x_\sigma$. Define $g : R_\sigma \to R_e$ by $g(r) = ry$ for any $r \in R_\sigma$. Then clearly $f = f \circ g \circ f$.
(2) It follows by using Theorem 4.2. ∎

Corollary 4.4 *Let G be a finite group and $R_\sigma = \oplus_{\sigma \in G} R_\sigma$ be a graded ring. Then R is von Neumann gr-regular if and only if the smash product $R\#G$ is a von Neumann regular ring.*

Proof: Assume that R is von Neumann gr-regular. Let $U = \oplus_{\sigma \in G} R(\sigma)$. Since R is a finitely generated object in $R - gr$, Theorem 4.2 shows that U is R-regular in $R - gr$. Since the σ-suspension functor $T_\sigma : R - gr \to R - gr$, $T_\sigma(M) = M(\sigma)$ for any $M \in R - gr$, is an isomorphism of categories, we get that $U = U(\sigma)$ is $R(\sigma)$-regular for any $\sigma \in G$. Since G is finite, we see that U is $\oplus_{\sigma \in G} R(\sigma)$-regular, i.e. U is U-regular. Hence $End_{R-gr}(U)$ is a von Neumann regular ring. But $End_{R-gr}(U) \simeq R\#G$, so $R\#G$ is also von Neumann regular. The converse is straightforward. ∎

Corollary 4.5 *Let $R_\sigma = \oplus_{\sigma \in G} R_\sigma$ be a ring graded by the arbitrary group G. If R is von Neumann gr-regular, then $End_{R_e}(R)$ and $End_{R-gr}(U)$ are semiprime rings.*

Proof: By Corollary 4.4, R is R_e-regular, and by Corollary 4.5 we have that U is a regular object in $R - gr$. Now the result follows from Proposition 3.8. ∎

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